

# Large Amplitude Gravitational Waves

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## Abstract

We derive an asymptotic solution of the Einstein field equations which describes the propagation of a thin, large amplitude gravitational wave into a curved space-time. The resulting equations have the same form as the colliding plane wave equations without one of the usual constraint equations.

## 1 Introduction

Gravitational waves are one of the most important features of Einstein's general theory of relativity. The Einstein field equations are highly nonlinear, and a question of fundamental interest is how nonlinearity affects the propagation of gravitational waves.

Small amplitude gravitational waves are well described by the linearized Einstein equations which completely neglect nonlinear effects. Large amplitude unidirectional gravitational plane waves are described by the exact Brinkmann-Rosen solution of the vacuum Einstein equations [1, 2]. Despite the nonlinearity of the Einstein equations, a gravitational plane wave propagates into flat space-time without distortion, and there are no dynamic nonlinear effects.

The simplest situation in which nonlinear effects are significant is when a large amplitude gravitational wave propagates into a curved space-time. An important special case is when the space-time ahead of the wave is that

of a counter-propagating gravitational plane wave. The resulting space-time has a two-parameter family of spacelike isometries, and the metric is given by the exact colliding plane wave solution of the vacuum Einstein equations [3, 4, 5, 6]. In the case of more general space-times ahead of the wave, exact solutions do not exist.

In this paper, we derive an asymptotic solution of the Einstein equations which describes the propagation of a thin, large-amplitude, pulse-like gravitational wave into a general curved space-time. The solution applies provided that the metric varies much more rapidly inside the wave than on either side of the wave. As a result, the wave can be approximated locally by a nonlinear plane wave which is distorted as it propagates into the curved space-time. For plane-polarized waves, the asymptotic solution is given by equations (3.1), (3.6), (3.7), and (3.9)–(3.11) below. For non-polarized waves, the asymptotic solution is given by (3.1), (6.3), (6.5), and (6.8)–(6.11). The asymptotic equations consist of the colliding plane wave equations without one of the usual constraint equations. The colliding plane wave equations are therefore canonical equations for nonlinear gravitational waves, and they describe a much larger class of solutions than the ones with exact plane-wave symmetry.

The nonlinearity of the asymptotic equations may lead to the development of a space-time singularity. A plane gravitational wave propagating into flat space-time does not steepen. Consequently, the mechanism of singularity formation in gravitational waves differs from the nonlinear steepening of waves in quasilinear hyperbolic systems which leads to the formation of shocks. Instead, the singularity formation is caused by the mutual focusing of the gravitational wave and the curved space-time into which it propagates. A second nonlinear effect described by the asymptotic equations is the permanent distortion of space-time by the passage of a gravitational wave. A curved pulse generates a backscattered gravitational wave which propagates into the space-time behind it.

In Section 2, we briefly summarize the exact colliding plane wave solution of the Einstein equations. In Section 3, we give an overview of the asymptotic expansion. In Section 4, we write out expansions of the metric components, the connection coefficients, and the Ricci curvature components. In Section 5, we construct a coordinate system in which the metric adopts its simplest form. In Section 6, we complete the derivation of the asymptotic equations. In Section 7, we show that the same equations follow from an expansion of the variational principle for the Einstein equations. In Section 8, we explain how to derive boundary conditions for the asymptotic equations, and in Section 9 we consider some specific physical examples.

## 2 Colliding plane waves

The vacuum Einstein field equations imply that

$$\mathbf{Ricci} = 0, \quad (2.1)$$

where  $\mathbf{Ricci}$  is the Ricci tensor associated with the metric tensor  $\mathbf{g}$ . The plane-polarized, colliding plane wave solution of (2.1) is given by

$$\mathbf{g} = -2e^{-M} du dv + e^{-U} (e^V dy^2 + e^{-V} dz^2), \quad (2.2)$$

where the functions  $M(u, v)$ ,  $U(u, v)$ ,  $V(u, v)$  satisfy the colliding plane wave equations,

$$U_{uv} = U_u U_v, \quad (2.3)$$

$$V_{uv} = \frac{1}{2} (U_u V_v + U_v V_u), \quad (2.4)$$

$$M_{uv} = \frac{1}{2} (-U_u U_v + V_u V_v), \quad (2.5)$$

$$U_{uu} = \frac{1}{2} (U_u^2 + V_u^2) - U_u M_u, \quad (2.6)$$

$$U_{vv} = \frac{1}{2} (U_v^2 + V_v^2) - U_v M_v. \quad (2.7)$$

Equations (2.3)–(2.5) are wave equations for  $M$ ,  $U$  and  $V$  in characteristic coordinates  $(u, v)$ . Equations (2.6)–(2.7) are constraints which are preserved by (2.3)–(2.5). To specify a unique solution, the wave equations can be supplemented by characteristic initial data for  $M$ ,  $U$ ,  $V$  on the lines  $u = 0$  and  $v = 0$  which satisfy the appropriate constraint equations.

The metric which describes the collision of non-polarized plane waves is

$$\mathbf{g} = -2e^{-M} du dv + e^{-U} (e^V \cosh W dy^2 - 2 \sinh W dy dz + e^{-V} \cosh W dz^2),$$

where the functions  $M(u, v)$ ,  $U(u, v)$ ,  $V(u, v)$ ,  $W(u, v)$  satisfy

$$U_{uv} = U_u U_v, \quad (2.8)$$

$$V_{uv} = \frac{1}{2} (U_u V_v + U_v V_u) - (V_u W_v + V_v W_u) \tanh W, \quad (2.9)$$

$$W_{uv} = \frac{1}{2} (U_u W_v + U_v W_u) + V_u V_v \sinh W \cosh W, \quad (2.10)$$

$$M_{uv} = \frac{1}{2} (-U_u U_v + V_u V_v \cosh^2 W + W_u W_v), \quad (2.11)$$

$$U_{uu} = \frac{1}{2} (U_u^2 + V_u^2 \cosh^2 W + W_u^2) - U_u M_u, \quad (2.12)$$

$$U_{vv} = \frac{1}{2} (U_v^2 + V_v^2 \cosh^2 W + W_v^2) - U_v M_v. \quad (2.13)$$

When  $W = 0$ , this solution reduces to the plane-polarized solution. When all functions are independent of  $v$ , the solution reduces to the Rosen form of the exact unidirectional plane wave solution.

### 3 Overview of the expansion

In this section, we outline the main ideas of the derivation of the asymptotic solution. For simplicity, we describe the case of plane-polarized waves. The algebraic details are given in the following sections.

We consider metrics of the form

$$\begin{aligned} \mathbf{g} &= \mathbf{g} \left( \frac{u(x)}{\varepsilon}, x; \varepsilon \right), \\ \mathbf{g}(\theta, x; \varepsilon) &= \mathbf{g}^0(\theta, x) + \varepsilon \mathbf{g}^1(\theta, x) + O(\varepsilon^2), \end{aligned} \quad (3.1)$$

where  $\varepsilon$  is a small parameter and  $u$  is a scalar-valued phase function with  $du \neq 0$ . This ansatz corresponds to a metric that varies rapidly and strongly in the  $u$ -direction. The phase  $u$  is a null function of the metric, at least up to the order  $\varepsilon$ . That is, it satisfies

$$\mathbf{g}^\sharp(du, du) = O(\varepsilon^2), \quad (3.2)$$

where  $\mathbf{g}^\sharp$  is the contravariant form of the metric tensor. The component form of this equation is written out in (4.5) below.

The scaled variable

$$\theta = \frac{u}{\varepsilon} \quad (3.3)$$

is a “stretched” coordinate inside the wave. We assume that the derivatives of  $\mathbf{g}(\theta, x; \varepsilon)$  with respect to  $\theta$  decay to zero sufficiently quickly as  $\theta \rightarrow \infty$ . Thus, the solution (3.1) represents a thin, pulse-like gravitational wave located near the null surface  $u = 0$ . For example, if the metric is independent of  $\theta$  when  $|\theta|$  is sufficiently large, then the solution represents a thin “sandwich” wave which separates slowly varying metrics on either side.

The Ricci tensor associated with the metric (3.1) has an expansion of the form

$$\mathbf{Ricci} = \frac{1}{\varepsilon^2} \mathbf{Ricci}^{-2} + \frac{1}{\varepsilon} \mathbf{Ricci}^{-1} + O(1). \quad (3.4)$$

At leading order in  $\varepsilon$ , the Einstein equations (2.1) imply that

$$\mathbf{Ricci}^{-2} = 0.$$

This equation is a nonlinear, second order ordinary differential equation in  $\partial_\theta$  for the leading order term of the metric in which the “slow” variables  $x$  occur as parameters. We write it symbolically as

$$N(\partial_\theta^2) \begin{bmatrix} 0 \\ \mathbf{g} \end{bmatrix} = 0. \quad (3.5)$$

In suitable coordinates  $(u, v, y, z)$ , a solution of this equation is the plane-polarized plane wave metric

$$\overset{0}{\mathbf{g}} = -2e^{-M} du dv + e^{-U} (e^V dy^2 + e^{-V} dz^2), \quad (3.6)$$

where  $M, U, V$  are functions of  $(\theta, v, y, z)$ . For a metric of the form (3.6), equation (3.5) reduces to the  $\theta$ -constraint equation,

$$U_{\theta\theta} = \frac{1}{2} (U_\theta^2 + V_\theta^2) - U_\theta M_\theta. \quad (3.7)$$

At the next order in  $\varepsilon$ , the Einstein equations imply that

$$\overset{-1}{\mathbf{Ricci}} = 0.$$

This is a linear equation for  $\overset{1}{\mathbf{g}}$  of the form

$$L(\partial_\theta^2) \begin{bmatrix} 1 \\ \mathbf{g} \end{bmatrix} = F(\partial_\theta, \partial_v, \partial_y, \partial_z) \begin{bmatrix} 0 \\ \mathbf{g} \end{bmatrix}, \quad (3.8)$$

where  $L$  is a second order linear ordinary differential operator in  $\partial_\theta$  acting on  $\overset{1}{\mathbf{g}}$ , with coefficients depending on  $\overset{0}{\mathbf{g}}$ , and  $F$  is a nonlinear partial differential operator acting on  $\overset{0}{\mathbf{g}}$ . The equations in (3.8) are not independent. The requirement that (3.8) can be solved for  $\overset{1}{\mathbf{g}}$  implies that  $M, U$ , and  $V$  satisfy the equations

$$U_{\theta v} = U_\theta U_v, \quad (3.9)$$

$$V_{\theta v} = \frac{1}{2} (U_\theta V_v + U_v V_\theta), \quad (3.10)$$

$$M_{\theta v} = \frac{1}{2} (-U_\theta U_v + V_\theta V_v). \quad (3.11)$$

Equations (3.9)–(3.11) are identical to the evolution equations (2.8)–(2.10) for the exact colliding plane wave solution, with  $\theta = u/\varepsilon$ . The leading order solution satisfies the constraint equation (3.7) in the “fast” phase variable  $\theta$ ,

but it need not satisfy the constraint equation (2.7) in the “slow” variable  $v$ . If the  $v$ -constraint equation does not hold, then the asymptotic expansion of the metric contains higher order terms which are absent in the exact colliding plane wave solution.

Equation (3.6) implies that  $\partial_v = -e^M \mathbf{g}^\# \cdot du$ . Thus,  $\partial_v$  is a vector on the light cone which is tangent to the null surface  $u = 0$ , and the “slow” derivative with respect to  $v$  which appears in (3.9)–(3.11) is a derivative along the bicharacteristic null geodesics associated with  $u$ . The transverse variables  $y$  and  $z$  occur as parameters. Therefore, in the short-wave limit considered here, the  $(1 + 3)$ -dimensional field equations reduce to  $(1 + 1)$ -dimensional asymptotic equations along the set of null geodesics associated with the phase  $u$ . The parametric dependence of the solution on  $y$  and  $z$  allows the pulse to be compactly supported in the transverse directions, so that the wave need not have infinite extent. Moreover, the asymptotic solution need not have any special exact symmetries.

The asymptotic equations for non-polarized gravitational waves are obtained in a similar way. They consist of the general colliding plane wave equations (2.8)–(2.12) with  $u$  replaced by  $\theta$ . The  $v$ -constraint equation (2.13) is not required to hold.

Since the asymptotic equations follow from the order  $\varepsilon^{-2}$  and order  $\varepsilon^{-1}$  components of the field equations, the asymptotic solution remains valid in the presence of matter with a slowly varying, order one energy-momentum tensor,  $\mathbf{T} = \mathbf{T}(x)$ .

One subtle point in carrying out the expansion concerns the choice of the phase function  $u$ . In order for (3.5) to have a nontrivial solution, the phase  $u$  must be a null function of the leading order metric, but  $u$  need not be a null function of the entire metric. However, it follows from the analysis in Section 5 that we can use a transformation of the form

$$u \rightarrow \varepsilon \Psi \left( \frac{u}{\varepsilon}, x; \varepsilon \right) \quad (3.12)$$

to choose a phase which satisfies (3.2). The asymptotic solutions obtained with the use of the old and the new phases can be shown to be equivalent. When the phase satisfies (3.2), variations in the metric propagate along the null geodesics associated with the phase, and the asymptotic equations adopt their simplest form.

## 4 Expansion of the metric and the curvature

In this section, we write out expansions of the metric components, the connection coefficients, and the Ricci curvature components.

We use local coordinates  $x^\alpha$  in which

$$\mathbf{g} = g_{\alpha\beta} dx^\alpha dx^\beta. \quad (4.1)$$

Here and below, Greek indices  $\alpha, \beta, \mu, \nu, \dots$  take on the values  $0, 1, 2, 3$ . We look for an expansion of the metric components as  $\varepsilon \rightarrow 0$  of the form

$$\begin{aligned} g_{\alpha\beta} &= g_{\alpha\beta} \left( \frac{u(x)}{\varepsilon}, x; \varepsilon \right), \\ g_{\alpha\beta}(\theta, x; \varepsilon) &= g_{\alpha\beta}^0(\theta, x) + \varepsilon g_{\alpha\beta}^1(\theta, x) + O(\varepsilon^2). \end{aligned} \quad (4.2)$$

The contravariant metric components  $g^{\alpha\beta}$  satisfy

$$g^{\alpha\mu} g_{\mu\beta} = \delta_\beta^\alpha.$$

Expansion of this equation in a power series in  $\varepsilon$  gives

$$g^{\alpha\beta} = g^{\alpha\beta}_0 - \varepsilon g^{\alpha\beta}_1 + O(\varepsilon^2). \quad (4.3)$$

In (4.3),  $g^{\alpha\beta}_0$  is the inverse of  $g_{\alpha\beta}^0$ , and we use the leading order metric components to raise indices, so that

$$g^{\alpha\beta}_1 = g^{\alpha\mu}_0 g^{\beta\nu}_0 g_{\mu\nu}^1. \quad (4.4)$$

With this notation, the order  $\varepsilon$  term in the expansion of the contravariant metric component  $g^{\alpha\beta}$  is  $-g^{\alpha\beta}_1$ , not  $g^{\alpha\beta}_1$ .

In terms of the metric components, we have

$$\begin{aligned} \mathbf{g}^\#(du, du) &= g^{\alpha\beta} \frac{\partial u}{\partial x^\alpha} \frac{\partial u}{\partial x^\beta} \\ &= g^{\alpha\beta}_0 \frac{\partial u}{\partial x^\alpha} \frac{\partial u}{\partial x^\beta} - \varepsilon g^{\alpha\beta}_1 \frac{\partial u}{\partial x^\alpha} \frac{\partial u}{\partial x^\beta} + O(\varepsilon^2). \end{aligned} \quad (4.5)$$

Thus, the null condition (3.2) holds provided that

$$g^{\alpha\beta}_0 \frac{\partial u}{\partial x^\alpha} \frac{\partial u}{\partial x^\beta} = 0, \quad g^{\alpha\beta}_1 \frac{\partial u}{\partial x^\alpha} \frac{\partial u}{\partial x^\beta} = 0. \quad (4.6)$$

The first condition in (4.6) states that  $u$  is a null function of  $\overset{0}{\mathbf{g}}$ . The second condition implies that the phase is a null function of the perturbed metric, at least up to the first order in  $\varepsilon$ .

The covariant components  $R_{\alpha\beta}$  of the Ricci tensor are given by

$$R_{\alpha\beta} = \frac{\partial \Gamma_{\alpha\beta}^{\lambda}}{\partial x^{\lambda}} - \frac{\partial \Gamma_{\beta\lambda}^{\lambda}}{\partial x^{\alpha}} + \Gamma_{\alpha\beta}^{\lambda} \Gamma_{\lambda\mu}^{\mu} - \Gamma_{\alpha\lambda}^{\mu} \Gamma_{\beta\mu}^{\lambda}, \quad (4.7)$$

where  $\Gamma_{\alpha\beta}^{\lambda}$  are the connection coefficients

$$\Gamma_{\alpha\beta}^{\lambda} = \frac{1}{2} g^{\lambda\mu} \left( \frac{\partial g_{\beta\mu}}{\partial x^{\alpha}} + \frac{\partial g_{\alpha\mu}}{\partial x^{\beta}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\mu}} \right). \quad (4.8)$$

From (3.3), the derivative of a function  $f_{\alpha\beta}(\theta, x)$ , with respect to  $x^{\mu}$  is given by

$$\frac{\partial f_{\alpha\beta}}{\partial x^{\mu}} = \frac{1}{\varepsilon} f_{\alpha\beta,\theta} u_{\mu} + f_{\alpha\beta,\mu}, \quad (4.9)$$

where

$$u_{\mu} = \frac{\partial u}{\partial x^{\mu}}, \quad f_{\alpha\beta,\theta} = \frac{\partial f_{\alpha\beta}}{\partial \theta} \Big|_x, \quad f_{\alpha\beta,\mu} = \frac{\partial f_{\alpha\beta}}{\partial x^{\mu}} \Big|_{\theta}.$$

We use (4.2), (4.3), and (4.9) in (4.7) and (4.8) and expand the result with respect to  $\varepsilon$ . After some algebra, we find that

$$\begin{aligned} \Gamma_{\alpha\beta}^{\lambda} &= \frac{1}{\varepsilon} \overset{-1}{\Gamma}_{\alpha\beta}^{\lambda} + \overset{0}{\Gamma}_{\alpha\beta}^{\lambda} + O(\varepsilon), \\ R_{\alpha\beta} &= \frac{1}{\varepsilon^2} \overset{-2}{R}_{\alpha\beta} + \frac{1}{\varepsilon} \overset{-1}{R}_{\alpha\beta} + O(1), \end{aligned} \quad (4.10)$$

where

$$\begin{aligned} \overset{-1}{\Gamma}_{\alpha\beta}^{\lambda} &= \frac{1}{2} \overset{0}{g}^{\lambda\mu} \left( \overset{0}{g}_{\beta\mu,\theta} u_{\alpha} + \overset{0}{g}_{\alpha\mu,\theta} u_{\beta} - \overset{0}{g}_{\alpha\beta,\theta} u_{\mu} \right), \\ \overset{0}{\Gamma}_{\alpha\beta}^{\lambda} &= \frac{1}{2} \overset{0}{g}^{\lambda\mu} \left( \overset{0}{g}_{\beta\mu,\alpha} + \overset{0}{g}_{\alpha\mu,\beta} - \overset{0}{g}_{\alpha\beta,\mu} \right) \\ &\quad + \frac{1}{2} \overset{0}{g}^{\lambda\mu} \left( \overset{1}{g}_{\beta\mu,\theta} u_{\alpha} + \overset{1}{g}_{\alpha\mu,\theta} u_{\beta} - \overset{1}{g}_{\alpha\beta,\theta} u_{\mu} \right) \\ &\quad - \frac{1}{2} \overset{1}{g}^{\lambda\mu} \left( \overset{0}{g}_{\beta\mu,\theta} u_{\alpha} + \overset{0}{g}_{\alpha\mu,\theta} u_{\beta} - \overset{0}{g}_{\alpha\beta,\theta} u_{\mu} \right), \\ \overset{-2}{R}_{\alpha\beta} &= \overset{-1}{\Gamma}_{\alpha\beta,\theta}^{\mu} u_{\mu} - \overset{-1}{\Gamma}_{\beta\mu,\theta}^{\mu} u_{\alpha} + \overset{-1}{\Gamma}_{\alpha\beta}^{\mu} \overset{-1}{\Gamma}_{\mu\nu}^{\nu} - \overset{-1}{\Gamma}_{\alpha\nu}^{\mu} \overset{-1}{\Gamma}_{\beta\mu}^{\nu}, \\ \overset{-1}{R}_{\alpha\beta} &= \overset{-1}{\Gamma}_{\alpha\beta,\mu}^{\mu} - \overset{-1}{\Gamma}_{\beta\mu,\alpha}^{\mu} + \overset{0}{\Gamma}_{\alpha\beta,\theta}^{\mu} u_{\mu} - \overset{0}{\Gamma}_{\beta\mu,\theta}^{\mu} u_{\alpha} \\ &\quad + \overset{-1}{\Gamma}_{\alpha\beta}^{\mu} \overset{0}{\Gamma}_{\mu\nu}^{\nu} + \overset{0}{\Gamma}_{\alpha\beta}^{\mu} \overset{-1}{\Gamma}_{\mu\nu}^{\nu} - \overset{-1}{\Gamma}_{\alpha\nu}^{\mu} \overset{0}{\Gamma}_{\beta\mu}^{\nu} - \overset{0}{\Gamma}_{\alpha\nu}^{\mu} \overset{-1}{\Gamma}_{\beta\mu}^{\nu}. \end{aligned} \quad (4.11)$$

The component form of the field equations (2.1) is

$$R_{\alpha\beta} = 0. \quad (4.12)$$

Using (4.10) in (4.12) and equating coefficients of  $\varepsilon^{-2}$  and  $\varepsilon^{-1}$  to zero, we get that

$$\overset{-2}{R}_{\alpha\beta} = 0, \quad (4.13)$$

$$\overset{-1}{R}_{\alpha\beta} = 0. \quad (4.14)$$

In order to solve these equations, we first use a coordinate transformation to simplify the form of the metric.

## 5 Coordinate transformations

In this section, we show that there is a choice of a local coordinate system  $x^\alpha$  in which  $u = x^0$  and the metric has the form

$$\begin{aligned} \mathbf{g} = & 2 \overset{0}{g}_{01} dx^0 dx^1 + \overset{0}{g}_{ab} dx^a dx^b \\ & + \varepsilon \left\{ 2 \overset{1}{g}_{1a} dx^1 dx^a + \overset{1}{g}_{ab} dx^a dx^b \right\} + O(\varepsilon^2). \end{aligned} \quad (5.1)$$

Here and below, indices  $a, b, c, \dots$  take on the values 2, 3, while indices  $i, j, k, \dots$  take on the values 1, 2, 3.

The corresponding expansion of the contravariant form of the metric tensor is

$$\begin{aligned} \mathbf{g}^\sharp = & 2 \overset{0}{g}^{01} \partial_0 \partial_1 + \overset{0}{g}^{ab} \partial_a \partial_b \\ & - \varepsilon \left\{ 2 \overset{0}{g}^{01} \overset{0}{g}^{ab} \overset{1}{g}_{1b} \partial_0 \partial_a + \overset{0}{g}^{ac} \overset{0}{g}^{bd} \overset{1}{g}_{cd} \partial_a \partial_b \right\} + O(\varepsilon^2). \end{aligned} \quad (5.2)$$

For this metric, we have

$$\overset{0}{g}^{00} = 0, \quad \overset{1}{g}^{00} = 0. \quad (5.3)$$

Thus, the phase  $u = x^0$  satisfies (4.6), and hence (3.2).

The most general coordinate transformation which is compatible with the expansion (4.2) has the form

$$\frac{x^0}{\varepsilon} \rightarrow \overset{1}{\Psi}^0 \left( \frac{x^0}{\varepsilon}, x \right) + \varepsilon \overset{2}{\Psi}^0 \left( \frac{x^0}{\varepsilon}, x \right) + O(\varepsilon^2), \quad (5.4)$$

$$x^i \rightarrow \overset{0}{\Psi}^i(x) + \varepsilon \overset{1}{\Psi}^i \left( \frac{x^0}{\varepsilon}, x \right) + \varepsilon^2 \overset{2}{\Psi}^i \left( \frac{x^0}{\varepsilon}, x \right) + O(\varepsilon^3). \quad (5.5)$$

We suppose that the phase is given by  $u = x^0$  in both the old and the new coordinates. Thus, the change of coordinates (5.4) implies a change in the phase of the form (3.12).

First we simplify the leading order metric components by means of a transformation of the form

$$x^0 \rightarrow x^0, \quad x^i \rightarrow x^i + \varepsilon \Psi^i \left( \frac{x^0}{\varepsilon}, x \right). \quad (5.6)$$

Expansion of the transformation law for the change in covariant tensor components implies that the leading order metric components transform under (5.6) according to

$$g_{00}^0 \rightarrow g_{00}^0 + 2 \Psi_{,\theta}^k g_{0k}^0 + \Psi_{,\theta}^k \Psi_{,\theta}^l g_{kl}^0, \quad (5.7)$$

$$g_{0i}^0 \rightarrow g_{0i}^0 + \Psi_{,\theta}^k g_{ki}^0, \quad (5.8)$$

$$g_{ij}^0 \rightarrow g_{ij}^0. \quad (5.9)$$

If the matrix  $g_{ij}^0$  is non-singular, then (5.8) implies that we can transform  $g_{0i}^0$  to zero. This contradicts the requirement that  $x^0$  is null (cf. [2], Section 109). Hence, we must have

$$\det g_{ij}^0 = 0. \quad (5.10)$$

By an appropriate renumbering of the  $i$ -coordinates, we can suppose without loss of generality that

$$\det g_{ab}^0 \neq 0. \quad (5.11)$$

From (5.7) and (5.8), we can then choose the transformation (5.6) so that

$$g_{00}^0 = g_{02}^0 = g_{03}^0 = 0. \quad (5.12)$$

Solving equation (5.10) for  $g_{11}^0$ , we get

$$g_{11}^0 = g^{ab} g_{1a}^0 g_{1b}^0, \quad (5.13)$$

where  $g^{ab}$  is the inverse of  $g_{ab}^0$ . We define

$$g^a = g^{ab} g_{1b}^0. \quad (5.14)$$

From (5.13)–(5.14), it follows that

$$\overset{0}{g}_{11} = \overset{0}{g}_{cd} g^c g^d, \quad \overset{0}{g}_{1a} = \overset{0}{g}_{ac} g^c. \quad (5.15)$$

Using (5.12)–(5.15) in (4.1), we find that in the transformed coordinate system, the metric has the form

$$\mathbf{g} = 2 \overset{0}{g}_{01} dx^0 dx^1 + \overset{0}{g}_{ab} (dx^a + g^a dx^1) (dx^b + g^b dx^1) + O(\varepsilon). \quad (5.16)$$

From (4.13), the metric (5.16) must satisfy the condition

$$\overset{-2}{R}_{ab} = 0. \quad (5.17)$$

Using (5.16) in (4.11), we find that

$$\overset{-2}{R}_{ab} = -\frac{1}{2} (\overset{0}{g}^{01})^2 \overset{0}{g}_{ac} g_{,\theta}^c \overset{0}{g}_{bd} g_{,\theta}^d. \quad (5.18)$$

Equations (5.17)–(5.18) imply that

$$g_{,\theta}^a = 0,$$

so  $g^a$  is independent of  $\theta$ . This fact allows us to remove  $g^a$  by a transformation

$$x^a \rightarrow \Psi^a(x^1, x^c). \quad (5.19)$$

The form of the metric (5.16) is unchanged by (5.19), and

$$\begin{aligned} \overset{0}{g}_{ab} &\rightarrow \Psi_{,a}^c \Psi_{,b}^d \overset{0}{g}_{cd}, \\ g^a &\rightarrow (A^{-1})_c^a (g^c + \Psi_{,1}^c), \end{aligned} \quad (5.20)$$

where  $(A_c^a) = (\Psi_{,c}^a)$ . From (5.20), we can set  $g^a = 0$ . The metric (5.16) then reduces to

$$\mathbf{g} = 2 \overset{0}{g}_{01} dx^0 dx^1 + \overset{0}{g}_{ab} dx^a dx^b + O(\varepsilon). \quad (5.21)$$

Next, we simplify the form of  $\overset{1}{\mathbf{g}}$ . We consider the transformation of coordinates

$$x^0 \rightarrow \varepsilon \overset{1}{\Psi}^0 \left( \frac{x^0}{\varepsilon}, x \right) + \varepsilon^2 \overset{2}{\Psi}^0 \left( \frac{x^0}{\varepsilon}, x \right), \quad x^i \rightarrow x^i + \varepsilon^2 \overset{2}{\Psi}^i \left( \frac{x^0}{\varepsilon}, x \right). \quad (5.22)$$

Under the action of (5.22), the form (5.21) of the metric is unchanged at order zero and

$${}^0g_{01} \rightarrow {}^1\Psi_{,\theta} {}^0g_{01} .$$

At order one, the components transform according to

$$\begin{aligned} {}^1g_{00} &\rightarrow {}^1\Psi_{,\theta} ({}^1\Psi_{,\theta} {}^1g_{00} + 2 {}^2\Psi_{,\theta} {}^1g_{01}), \\ {}^1g_{01} &\rightarrow {}^1\Psi_{,\theta} {}^1g_{01} + ({}^1\Psi_{,0} + {}^2\Psi_{,\theta}) {}^0g_{01}, \\ {}^1g_{0a} &\rightarrow {}^1\Psi_{,\theta} {}^1g_{0a} + {}^2\Psi_{,\theta}^b {}^0g_{ab}, \\ {}^1g_{11} &\rightarrow {}^1g_{11} + 2 {}^1\Psi_{,1} {}^0g_{01}, \\ {}^1g_{1a} &\rightarrow {}^1g_{1a} + {}^1\Psi_{,a} {}^0g_{01}, \\ {}^1g_{ab} &\rightarrow {}^1g_{ab} . \end{aligned}$$

These transformations can be used to make

$${}^1g_{11} = {}^1g_{0\alpha} = 0. \quad (5.23)$$

The resulting metric then has the form given in (5.1).

Use of (5.1) and (5.2) in (4.11) implies that the nonzero connection coefficients at the orders  $\varepsilon^{-1}$  and  $\varepsilon^0$  are

$$\begin{aligned} {}^{-1}\Gamma^0_{00} &= {}^0g^{01} {}^0g_{01,\theta}, \quad {}^{-1}\Gamma^1_{ab} = -\frac{1}{2} {}^0g^{01} {}^0g_{ab,\theta}, \quad {}^{-1}\Gamma^a_{0b} = \frac{1}{2} {}^0g^{ac} {}^0g_{bc,\theta}, \\ {}^0\Gamma^0_{00} &= {}^0g^{01} {}^0g_{01,0}, \quad {}^0\Gamma^0_{0a} = \frac{1}{2} {}^0g^{01} ({}^0g_{01,a} + {}^1g_{1a,\theta}) - \frac{1}{2} {}^1g^{0b} {}^0g_{ab,\theta}, \\ {}^0\Gamma^0_{ab} &= -\frac{1}{2} {}^0g^{01} {}^0g_{ab,1}, \quad {}^0\Gamma^1_{11} = {}^0g^{01} {}^0g_{01,1}, \quad {}^0\Gamma^1_{1a} = \frac{1}{2} {}^0g^{01} ({}^0g_{01,a} - {}^1g_{1a,\theta}), \\ {}^0\Gamma^1_{ab} &= -\frac{1}{2} {}^0g^{01} ({}^0g_{ab,0} + {}^1g_{ab,\theta}), \quad {}^0\Gamma^a_{01} = -\frac{1}{2} {}^0g^{ac} ({}^0g_{01,c} - {}^1g_{1c,\theta}), \\ {}^0\Gamma^a_{0b} &= \frac{1}{2} {}^0g^{ac} ({}^0g_{bc,0} + {}^1g_{bc,\theta}) - \frac{1}{2} {}^1g^{ac} {}^0g_{bc,\theta}, \\ {}^0\Gamma^a_{1b} &= \frac{1}{2} {}^0g^{ac} {}^0g_{bc,1}, \quad {}^0\Gamma^a_{bc} = \frac{1}{2} {}^0g^{ad} ({}^0g_{bd,c} + {}^0g_{cd,b} - {}^0g_{bc,d}) + \frac{1}{2} {}^1g^{0a} {}^0g_{bc,\theta} . \end{aligned}$$

The nonzero components of the Ricci curvature at the orders  $\varepsilon^{-2}$  and  $\varepsilon^{-1}$  are

$${}^{-2}R_{00} = -\frac{1}{2} ({}^0g^{ab} {}^0g_{ab,\theta})_{,\theta} - \frac{1}{4} {}^0g^{ac} {}^0g_{bc,\theta} {}^0g^{bd} {}^0g_{ad,\theta}$$

$$+\frac{1}{2}g^{01}g^{00}_{01,\theta}g^{0ab}g^{0ab}_{ab,\theta}, \quad (5.24)$$

$$R_{01}^{-1} = -(g^{01}g^{00}_{01,1})_{\theta} - \frac{1}{2}(g^{0ab}g^{0ab}_{ab,1})_{\theta} - \frac{1}{4}g^{0ac}g^{0bc}_{bc,\theta}g^{0bd}g^{0bd}_{ad,1}, \quad (5.25)$$

$$R_{ab}^{-1} = -g^{01}\left(g^{0ab}_{ab,1\theta} - \frac{1}{2}g^{cd}(g^{0ac}_{ac,\theta}g^{0bd}_{bd,1} + g^{0ac}_{ac,1}g^{0bd}_{bd,\theta})\right. \\ \left. + \frac{1}{4}g^{cd}(g^{0cd}_{cd,1}g^{0ab}_{ab,\theta} + g^{0cd}_{cd,\theta}g^{0ab}_{ab,1})\right). \quad (5.26)$$

$$R_{00}^{-1} = -\frac{1}{2}g^a_{a,\theta\theta} - \frac{1}{2}g^{0ac}g^{0bc}_{bc,\theta}g^b_{a,\theta} + \frac{1}{2}g^{01}g^{00}_{01,\theta}g^a_{a,\theta} - (g^{0ab}g^{0ab}_{ab,\theta})_0 \\ - \frac{1}{2}g^{0ac}g^{0bc}_{bc,\theta}g^{0bd}g^{0bd}_{ad,0} + \frac{1}{2}g^{01}g^{0ab}(g^{00}_{01,\theta}g^{0ab}_{ab,0} + g^{00}_{01,0}g^{0ab}_{ab,\theta}) \quad (5.27)$$

$$R_{0a}^{-1} = \frac{1}{2}(g^{0ab}g^{01}(g^{00}_{01}g^{00b})_{\theta})_{\theta} + \frac{1}{4}g^{cd}g^{0cd}_{cd,\theta}g^{0ab}g^{01}(g^{00}_{01}g^{00b})_{\theta} \\ - \frac{1}{2}(g^{01}g^{00}_{01,a} + g^{0cd}g^{0cd}_{cd,a})_{\theta} + \frac{1}{2}(g^{0bc}g^{0ab}_{ab,\theta})_c + \frac{1}{4}g^{0bc}g^{0ab}_{ab,\theta}g^{0de}g^{0de}_{de,c} \\ + \frac{1}{4}g^{01}g^{00}_{01,a}g^{0cd}g^{0cd}_{cd,\theta} - \frac{1}{4}g^{0bd}g^{0cd}_{cd,\theta}g^{0ce}g^{0ce}_{be,a}, \quad (5.28)$$

## 6 The asymptotic expansion

We choose coordinates

$$(x^0, x^1, x^2, x^3) = (u, v, y, z) \quad (6.1)$$

in which the metric has the form (5.1). We introduce functions  $M$ ,  $U$ ,  $V$ ,  $W$  of  $(\theta, v, y, z)$  such that

$$\begin{aligned} g^{00}_{01} &= -e^{-M}, \\ \begin{pmatrix} g^{00}_{01} \\ g^{0ab} \end{pmatrix} &= \begin{pmatrix} e^{-U+V} \cosh W & -e^{-U} \sinh W \\ -e^{-U} \sinh W & e^{-U-V} \cosh W \end{pmatrix}. \end{aligned} \quad (6.2)$$

It follows from (5.1), (6.1), and (6.2) that the leading order metric has the form of the colliding plane wave metric,

$$\begin{aligned} g^{00} &= -2e^{-M} du dv + e^{-U}(e^V \cosh W dy^2 - 2 \sinh W dy dz \\ &\quad + e^{-V} \cosh W dz^2). \end{aligned} \quad (6.3)$$

From (5.24), the only component of the leading order perturbation equation (4.13) which is not identically satisfied is

$$R_{00}^{-2} = 0. \quad (6.4)$$

Using (5.24) and (6.2) in (6.4), we obtain the  $\theta$ -constraint equation,

$$U_{\theta\theta} = \frac{1}{2} (U_\theta^2 + V_\theta^2 \cosh^2 W + W_\theta^2) - U_\theta M_\theta. \quad (6.5)$$

From (5.25)–(5.28), the only components of the first order perturbation equation (4.14) which are not identically satisfied are

$$\overset{-1}{R}_{01} = 0, \quad \overset{-1}{R}_{ab} = 0, \quad (6.6)$$

$$\overset{-1}{R}_{00} = 0, \quad \overset{-1}{R}_{0a} = 0. \quad (6.7)$$

Using (5.25)–(5.26) and (6.2) in (6.6), we get the evolution equations in the colliding plane wave equations,

$$U_{\theta v} = U_\theta U_v, \quad (6.8)$$

$$V_{\theta v} = \frac{1}{2} (U_\theta V_v + U_v V_\theta) - (V_\theta W_v + V_v W_\theta) \tanh W, \quad (6.9)$$

$$W_{\theta v} = \frac{1}{2} (U_\theta W_v + U_v W_\theta) + V_\theta V_v \sinh W \cosh W. \quad (6.10)$$

$$M_{\theta v} = \frac{1}{2} (-U_\theta U_v + V_\theta V_v \cosh^2 W + W_\theta W_v). \quad (6.11)$$

From (5.2) and (5.27)–(5.28), we find that (6.7) is satisfied by a suitable choice of the first order metric components  $\overset{1}{g}_{ab}, \overset{1}{g}_{1a}$ .

## 7 Variational principle

The variational principle for the vacuum Einstein field equations is

$$\begin{aligned} \delta S &= 0, \quad S = \int L d^4 x, \\ L &= R \sqrt{-\det g}, \end{aligned} \quad (7.1)$$

where  $R$  is the scalar curvature,

$$R = g^{\alpha\beta} R_{\alpha\beta}.$$

Using (4.2), (4.3), and (4.10) to expand the scalar curvature, we obtain

$$\begin{aligned} R &= \frac{1}{\varepsilon^2} \overset{-2}{R} + \frac{1}{\varepsilon} \overset{-1}{R} + O(1), \\ \overset{-2}{R} &= \overset{0}{g}{}^{\alpha\beta} \overset{-2}{R}_{\alpha\beta}, \\ \overset{-1}{R} &= \overset{0}{g}{}^{\alpha\beta} \overset{-1}{R}_{\alpha\beta} - \overset{1}{g}{}^{\alpha\beta} \overset{-2}{R}_{\alpha\beta}. \end{aligned} \quad (7.2)$$

For a metric of the form (5.21), we find that

$$\begin{aligned} \overset{-2}{R} &= 0, \\ \overset{-1}{R} &= \overset{0}{g}{}^{ab} \overset{-1}{R}_{ab} + 2 \overset{0}{g}{}^{01} \overset{-1}{R}_{01} - \overset{1}{g}{}^{00} \overset{-2}{R}_{00}. \end{aligned} \quad (7.3)$$

The only order one metric component which appears in (7.3) is

$$\lambda = - \overset{1}{g}{}^{00}.$$

In the derivation of the asymptotic equations, we used a coordinate system in which  $\lambda = 0$  — see (5.3). In the variational principle,  $\lambda$  acts as a Lagrange multiplier for the constraint equation, so we do not set it to zero until after we take variations.

We use (7.3) in (7.1), expand the result with respect to  $\varepsilon$ , and write the expanded Lagrangian in terms of  $\lambda$  and the functions  $M, U, V, W$ , defined in (6.2). This gives

$$L = \frac{1}{\varepsilon} \overset{-1}{L} + O(1),$$

with

$$\begin{aligned} \overset{-1}{L} &= \left\{ -2M_{\theta v} - 4U_{\theta v} + 3U_{\theta}U_v + V_{\theta}V_v \cosh^2 W + W_{\theta}W_v \right\} e^{-U} \\ &+ \lambda \left\{ U_{\theta\theta} - \frac{1}{2} (U_{\theta}^2 + V_{\theta}^2 \cosh^2 W + W_{\theta}^2) + U_{\theta}M_{\theta} \right\} e^{-M-U} \end{aligned} \quad (7.4)$$

We make the change of variables in the integration

$$d^4x = du dv dy dz = \varepsilon d\theta dv dy dz,$$

and neglect the integration with respect to the parametric variables  $(y, z)$ . The leading order asymptotic variational principle then becomes

$$\delta \overset{0}{S} = 0, \quad \overset{0}{S} = \int \overset{-1}{L} d\theta dv.$$

Variations of  $\overset{0}{S}$  with respect to the first order metric component  $\lambda$  give the constraint (6.5). Variations with respect to  $M, U, V, W$  give the evolution equations (6.8)–(6.11), after we set  $\lambda = 0$ . It is permissible to set  $\lambda = 0$  because the constraint is a gauge-type constraint which is preserved by the evolution equations.

## 8 Boundary conditions

In this section, we discuss the derivation of boundary conditions for the asymptotic equations. For simplicity, we consider a “sandwich” wave located near the null surface  $u = 0$  which varies rapidly in a thin strip

$$\theta_- \leq \frac{u}{\varepsilon} \leq \theta_+.$$

We denote the slowly varying metrics on either side of the wave by

$$\mathbf{g} = \begin{cases} \mathbf{g}_+ & \text{in } u > 0 \\ \mathbf{g}_- & \text{in } u < 0 \end{cases}. \quad (8.1)$$

We consider a coordinate patch around a point on the surface  $u = 0$  with local coordinates  $(u, v, y, z)$  chosen as in the derivation of the asymptotic solution. In order for the metric outside the wave to join continuously with the solution inside, we must have

$$\begin{aligned} \mathbf{g}_\pm \rightarrow & -2e^{-M_\pm} du dv + e^{-U_\pm} (e^{V_\pm} \cosh W_\pm dy^2 \\ & - 2 \sinh W_\pm dy dz + e^{-V_\pm} \cosh W_\pm dz^2), \end{aligned} \quad (8.2)$$

as  $u \rightarrow 0^\pm$ , where  $M_\pm, U_\pm, V_\pm, W_\pm$  are functions of  $(v, y, z)$ . From (6.3), (8.2), and the continuity of the metric, it follows that the solution of (6.8)–(6.11) must satisfy the characteristic boundary conditions,

$$M = M_\pm, \quad U = U_\pm, \quad V = V_\pm, \quad W = W_\pm, \quad \text{when } \theta = \theta_\pm. \quad (8.3)$$

This data need not satisfy the constraint (2.13).

The asymptotic equations must be supplemented by a condition which specifies the profile of the wave. For example, we can impose a characteristic initial condition

$$M = M_0, \quad U = U_0, \quad V = V_0, \quad W = W_0, \quad \text{when } v = 0, \quad (8.4)$$

where  $M_0, U_0, V_0, W_0$  are functions of  $(\theta, y, z)$  which satisfy the constraint (6.5). The characteristic initial data must also be compatible with the characteristic boundary data, meaning that

$$M_0(\theta_\pm, y, z) = M_\pm(0, y, z),$$

together with analogous conditions for the other variables.

Equations (6.8)–(6.11), the characteristic initial condition (8.4) on  $v = 0$ , and the characteristic boundary condition (8.3) on  $\theta = \theta_-$  form a well-posed

problem. Provided that the solution inside the wave is free of singularities, this problem has a unique solution. In particular, the solution at  $\theta = \theta_+$  is uniquely determined. Thus, in principle, the asymptotic equations, together with the characteristic initial data (8.4), determine a set of jump relations which connect the minus and plus metrics ahead of and behind the wave, respectively. If the metric ahead of the wave is known, then the jump conditions provide characteristic boundary conditions on  $u = 0$  for the space-time behind the wave. Together with a characteristic initial condition on  $v = 0$  and  $u > 0$ , for example, this gives a characteristic initial value problem [7] for the full field equations. This problem determines the slowly varying metric behind the wave (at least locally).

For instance, in the case of a plane polarized wave, the solution of (3.9) for  $U$  is [6]

$$U(\theta, v) = -\log[f(\theta) + g(v)]. \quad (8.5)$$

Here  $f$  and  $g$  are functions of integration, and we do not explicitly show the possible parametric dependence of the functions on  $(y, z)$ . From (8.3), (8.4), and (8.5) we have

$$f(\theta) + g(0) = e^{-U_0(\theta)}, \quad f(\theta_-) + g(v) = e^{-U_-(v)}.$$

The solution is nonsingular provided that  $f(\theta) + g(v) > 0$ .

It follows from (8.5) that the jump in  $U$  satisfies

$$e^{-U_+(v)} - e^{-U_-(v)} = e^{-U_0(\theta_+)} - e^{-U_0(\theta_-)}.$$

Use of (8.5) in (3.10) gives a linear wave equation for  $V$ ,

$$(f + g) V_{\theta v} = \frac{1}{2} (g_v V_\theta + f_\theta V_v).$$

Solution of this equation with the characteristic initial data  $V = V_0$  on  $v = 0$  and the characteristic boundary data  $V = V_-$  on  $\theta = \theta_-$  determines, in principle, the solution  $V = V_+$  on  $\theta = \theta_+$ . Finally, when  $W = 0$ , we define the  $v$ -constraint function  $G$  by

$$G = U_{vv} - \frac{1}{2} (U_v^2 + V_v^2) + U_v M_v. \quad (8.6)$$

It follows from (8.6) and (3.9)–(3.11) that

$$G_\theta = U_\theta G.$$

Integration of this equation with respect to  $\theta$  implies that

$$\log G_+(v) - \log G_-(v) = U_+(v) - U_-(v).$$

This equation provides a jump condition for  $M$ .

One difficulty which arises in the formulation of boundary conditions ahead of the wave is that the metric  $\mathbf{g}_-$  may not be given in a coordinate system which is compatible with the coordinate system used in the derivation of the asymptotic equations. It is then necessary to construct compatible coordinates  $(u, v, y, z)$ . The  $u$ -coordinate is the phase, so it is a null coordinate of the metric which can be found by solving an eikonal equation, subject to appropriate initial conditions. The  $v$ -coordinate is a null coordinate which is orthogonal to  $u$ , while the  $y$  and  $z$  coordinates parametrize the null geodesics on the surface  $u = v = 0$ .

If the gravitational wavefront  $u = 0$  forms a caustic, then the solution of the eikonal equation becomes multi-valued. When this happens, the local plane-wave approximation breaks down, and the asymptotic solution is not valid. However, the focusing at a caustic of the congruence of null geodesics associated with the phase does not necessarily imply the formation of a space-time singularity.

## 9 Examples

In this Section, we derive boundary conditions for the asymptotic equations which correspond to the propagation of a non-planar gravitational wave into Minkowski space-time, the exterior Schwarzschild space-time, and Robertson-Walker space-time. In each example, we consider the case of spherical waves, where the boundary data can be explicitly computed. In this paper, we do not attempt to explore the physical consequences of the asymptotic equations in any detail. Our aim here is simply to illustrate how to apply the asymptotic equations to specific physical problems.

### 9.1 Nonplanar wave propagation into Minkowski space-time

We suppose that the space-time ahead of the wave is flat. In inertial coordinates  $(t, \vec{x})$ , with  $t = x^0$  and  $\vec{x} = (x^1, x^2, x^3)$ , the metric is

$$\mathbf{g}_- = -dt^2 + d\vec{x}^2.$$

We consider a wave with phase

$$u = \frac{t - w(\vec{x})}{\sqrt{2}}.$$

The phase  $u$  is a null function of  $\mathbf{g}_-$  if

$$|\nabla w|^2 = 1,$$

where  $\nabla$  is the gradient with respect to  $\vec{x}$ . We define

$$v = \frac{t + w(\vec{x})}{\sqrt{2}},$$

and choose coordinates  $y(\vec{x})$ ,  $z(\vec{x})$  such that  $\nabla w$ ,  $\nabla y$ ,  $\nabla z$  are orthogonal. In the  $(u, v, y, z)$  coordinates, we have

$$\mathbf{g}_- = -2du dv + \frac{1}{|\nabla y|^2} dy^2 + \frac{1}{|\nabla z|^2} dz^2. \quad (9.1)$$

A comparison of (9.1) with (8.2) shows that the minus boundary data is given by

$$M_- = 0, \quad e^{-U_-} = \frac{1}{|\nabla y||\nabla z|_{u=0}}, \quad e^{-V_-} = \frac{|\nabla y|}{|\nabla z|_{u=0}}, \quad W_- = 0.$$

For example, in the case of an outgoing spherical wave, suitable coordinates are

$$u = \frac{t-r}{\sqrt{2}}, \quad v = \frac{t+r}{\sqrt{2}}, \quad y = \vartheta, \quad z = \varphi, \quad (9.2)$$

where  $(r, \vartheta, \varphi)$  are spherical polar coordinates and  $t > 0$ . In  $(u, v, y, z)$  coordinates, the flat space-time metric is

$$\mathbf{g}_- = -2dudv + \frac{1}{2}(u-v)^2(dy^2 + \sin^2 y dz^2).$$

Evaluation of this metric at  $u = 0$  and a comparison with (8.2) gives the minus boundary data

$$M_- = 0, \quad e^{-U_-} = \frac{1}{2}v^2 \sin y, \quad e^{-V_-} = \sin y, \quad W_- = 0,$$

where  $v > 0$ . In this case,  $M_-$ ,  $V_-$ , and  $W_-$  are independent of  $v$ , while  $U = U_-$  satisfies the equation

$$U_{vv} = \frac{1}{2}U_v^2.$$

Thus, the boundary data satisfies the  $v$ -constraint equation (2.13). The solution is therefore identical to an exact solution for the collision of outgoing and incoming spherical waves, with the additional possibility of a slow

parametric dependence on the polar angles  $(\vartheta, \varphi)$ . Some exact solutions for spherical wave propagation into flat space-time are constructed in [8].

For an incoming spherical wave, we use

$$u = \frac{t+r}{\sqrt{2}}, \quad v = \frac{t-r}{\sqrt{2}},$$

where  $t < 0$ . This leads to the same boundary data as in the case of an outgoing spherical wave, but with  $v < 0$  instead of  $v > 0$ .

## 9.2 Gravitational waves incident on a black hole

The exterior Schwarzschild metric is

$$\mathbf{g}_- = -adt^2 + \frac{1}{a}dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \quad (9.3)$$

where  $r > 2m$  and

$$a(r) = 1 - \frac{2m}{r}.$$

The contravariant metric tensor is

$$\mathbf{g}_-^\# = -\frac{1}{a}\partial_t^2 + a\partial_r^2 + \frac{1}{r^2}\left(\partial_\vartheta^2 + \frac{1}{\sin^2\vartheta}\partial_\varphi^2\right).$$

For simplicity, we consider an axially symmetric phase of the form

$$u = \frac{t - w(r, \vartheta)}{\sqrt{2}}.$$

The function  $u$  is null if  $w$  satisfies the eikonal equation

$$aw_r^2 + \frac{1}{r^2}w_\vartheta^2 = \frac{1}{a}.$$

We define the orthogonal null coordinate

$$v = \frac{t + w(r, \vartheta)}{\sqrt{2}},$$

and choose a coordinate  $y(r, \vartheta)$  whose gradient is orthogonal to the gradient of  $w(r, \vartheta)$ . In that case, we have

$$y_r = -\frac{hw_\vartheta}{r^2}, \quad y_\vartheta = haw_r,$$

where  $h(r, \vartheta)$  is a suitable integrating factor. We take  $z = \varphi$ . In  $(u, v, y, z)$  coordinates, the Schwarzschild metric (9.3) is given by

$$\mathbf{g}_- = -2a du dv + \frac{r_-^2}{h_-^2} dy^2 + r_-^2 \sin^2 \vartheta dz^2. \quad (9.4)$$

Inversion of the change of coordinates  $(t, r, \vartheta, \varphi) \mapsto (u, v, y, z)$  implies that  $r = r_-(v, y)$  and  $\vartheta = \vartheta_-(v, y)$  on  $u = 0$  for suitable functions  $r_-$  and  $\vartheta_-$ . A comparison of (9.4) with (8.2) implies that the boundary data is given by

$$e^{-M_-} = a_-, \quad e^{-U_-} = \frac{r_-^2 \sin \vartheta_-}{h_-}, \quad e^{-V_-} = h_- \sin \vartheta_-, \quad W_- = 0,$$

where  $a_- = a(r_-)$  and  $h_- = h(r_-, \vartheta_-)$ .

In the case of an incoming spherical wave incident on the black hole, suitable coordinates are

$$u = \frac{t + A(r)}{\sqrt{2}}, \quad v = \frac{t - A(r)}{\sqrt{2}}, \quad y = \vartheta, \quad z = \varphi, \quad (9.5)$$

where

$$A_r = \frac{1}{a}.$$

Integration of this equation implies that

$$A(r) = r + \log(r - 2m).$$

In  $(u, v, y, z)$  coordinates, the exterior Schwarzschild metric is

$$\mathbf{g}_- = -2a du dv + r^2 (dy^2 + \sin^2 y dz^2). \quad (9.6)$$

From (9.5), we have  $r = r_-(v)$  on  $u = 0$  where the function  $r_-(v)$  is the solution of

$$A(r_-) = -\frac{v}{\sqrt{2}}. \quad (9.7)$$

A comparison of (9.6) with (8.2) implies that the boundary data ahead of the incoming spherical wave is given by

$$e^{-M_-} = a_-, \quad e^{-U_-} = r_-^2 \sin y, \quad e^{-V_-} = \sin y, \quad W_- = 0. \quad (9.8)$$

Dropping the minus subscripts, we find that the constraint function  $G$  in (8.6) for the boundary data (9.8) is given by

$$G = 2 \left( \frac{a_v r_v}{a r} - \frac{r_{vv}}{r} \right).$$

Differentiation of (9.7) with respect to  $v$  implies that

$$r_v = -\frac{a}{\sqrt{2}}, \quad r_{vv} = -\frac{a_v}{\sqrt{2}}.$$

Use of this equation in the expression for  $G$  implies that  $G = 0$ . Thus, the boundary data (9.8) satisfies the  $v$ -constraint equation (2.13).

Numerical solutions of the interaction of a spherical gravitational wave with a black hole appear in [9].

### 9.3 Gravitational waves in a Robertson-Walker space-time

The Robertson-Walker metric is

$$\mathbf{g}_- = -dt^2 + \frac{1}{R^2} \left\{ \frac{1}{1 - kr^2} dr^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2) \right\}, \quad (9.9)$$

where  $R(t)$  is the scale factor, and  $k = -1, 0, 1$ .

As in the Schwarzschild example, we consider an axially symmetric phase for simplicity, given by

$$u = \frac{I(t) - w(r, \vartheta)}{\sqrt{2}},$$

where

$$I_t = R, \quad (1 - kr^2) w_r^2 + \frac{1}{r^2} w_\vartheta^2 = 1. \quad (9.10)$$

We define an orthogonal null coordinate  $v$  by

$$v = \frac{I(t) + w(r, \vartheta)}{\sqrt{2}}.$$

We choose a coordinate  $y(r, \vartheta)$  whose gradient is orthogonal to the gradient of  $w(r, \vartheta)$ , so that

$$y_r = -\frac{hw_\vartheta}{r^2 \sqrt{1 - kr^2}}, \quad y_\vartheta = h \sqrt{1 - kr^2} w_r,$$

where  $h(r, \vartheta)$  is a suitable integrating factor, and take  $z = \varphi$ . In  $(u, v, y, z)$  coordinates, the Robertson-Walker metric (9.9) is given by

$$\mathbf{g}_- = -\frac{2}{R^2} du dv + \frac{r^2}{h^2 R^2} dy^2 + \frac{r^2}{R^2} \sin^2 \vartheta dz^2. \quad (9.11)$$

A comparison of (9.11) with (8.2) implies that the boundary data is given by

$$e^{-M_-} = \frac{1}{R_-^2}, \quad e^{-U_-} = \frac{r_-^2 \sin \vartheta_-}{h_- R_-^2}, \quad e^{-V_-} = h_- \sin \vartheta_-, \quad W_- = 0,$$

where  $r = r_-(v, y)$ ,  $\vartheta = \vartheta_-(v, y)$ ,  $R = R_-(v, y)$ , and  $h = h_-(v, y)$  on  $u = 0$ .

For an outgoing spherical wave in a Robertson-Walker space-time, suitable coordinates are

$$u = \frac{I(t) - w(r)}{\sqrt{2}}, \quad v = \frac{I(t) + w(r)}{\sqrt{2}}, \quad y = \vartheta, \quad z = \varphi,$$

where

$$w_r = \frac{1}{\sqrt{1 - kr^2}}.$$

Integration of this equation implies that

$$w(r) = \begin{cases} \sin^{-1} r & \text{if } k = 1, \\ r & \text{if } k = 0, \\ \sinh^{-1} r & \text{if } k = -1. \end{cases}$$

The corresponding boundary data is given by

$$e^{-M_-} = \frac{1}{R_-^2}, \quad e^{-U_-} = \frac{r_-^2 \sin y}{R_-^2}, \quad e^{-V_-} = \sin y, \quad W_- = 0, \quad (9.12)$$

where  $t_-(v)$  and  $r_-(v)$  are given by

$$I(t_-) = \frac{v}{\sqrt{2}}, \quad r_- = \begin{cases} \sin(v/\sqrt{2}) & \text{if } k = 1, \\ v/\sqrt{2} & \text{if } k = 0, \\ \sinh(v/\sqrt{2}) & \text{if } k = -1, \end{cases} \quad (9.13)$$

and  $R_- = R(t_-)$ .

Dropping the minus subscripts, we find that the constraint function  $G$  in (8.6) for the boundary data (9.12) is given by

$$G = 2 \left( \frac{R_{vv}}{R} - \frac{r_{vv}}{r} \right). \quad (9.14)$$

From (9.10) and (9.13), we find that

$$r_{vv} = -\frac{1}{2}kr, \quad R_{vv} = \frac{RR_{tt} - R_t^2}{2R^3}.$$

Use of these expressions in (9.14) gives

$$G = \frac{RR_{tt} - R_t^2}{R^4} + k.$$

Thus, in general, the boundary data (9.12) does not satisfy the  $v$ -constraint equation (2.13).

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